S.-T. Yau College Student Mathematics Contests 2017

Analysis and Differential Equations

Individual

Please solve the following problems.

1. Suppose $f(x)$ is a positive and continuous function over $[a,b]$ and $g(x)$ is a positive and decreasing function over $[a,b]$. Prove that

$$\frac{\int_a^b xf(x)g(x)dx}{\int_a^b f(x)g(x)dx} \le \frac{\int_a^b xf(x)dx}{\int_a^b f(x)dx}.$$

2. Let E be a closed subset of $\mathbb{R}^n$ and

$$E_r = \{x \in \mathbb{R}^n : \ d(x,E) = r\}, \quad \text{for } r > 0.$$

Prove that E_r is measurable and is of Lebesgue measure zero.

3. (1). Prove that

$$w \longrightarrow z = \int_0^w (1-x^n)^{-\frac{2}{n}} dx$$

is a conformal mapping from the unit disk onto the interior of a regular n-gon Γ.

(2). Find an explicit formula for the boundary value problem of the Laplace equation over an n-gon Γ, where ϕ is a continuous function on $\partial\Gamma$:

$$\begin{cases} \Delta u = 0, \\ u|_{\partial\Gamma} = \phi. \end{cases}$$

4. Prove the uniqueness of the following problem for $u(x,t)$ of

$$\begin{cases} u_t = u_{xx}, & \text{in} \quad (-1,1)\times(0,+\infty), \\ u(\pm 1,t) = 1, & \text{for} \quad t>0, \\ u(x,0) = 0, & \text{for} \quad x \in (-1,1), \\ u \text{ is uniformly bounded.} & \qquad\qquad (*) \end{cases}$$

Prove that the last condition $(*)$ is necessary for the result.

5. Let $(v,u)(x,t)$ $(0 \le x \le 1, t \ge 0)$ with $v > 0$ for $0 \le x \le 1$, $t \ge 0$ be the C^∞ smooth solution of the following initial boundary value problem

$$\begin{cases} v_t - u_x = 0, \quad u_t + \left(\frac{1}{v^\gamma}\right)_x = \left(\frac{u_x}{v}\right)_x, \quad 0<x<1, \ t>0 \\ \left(\frac{u_x}{v} - \frac{1}{v^\gamma}\right)(0,t) = \left(\frac{u_x}{v} - \frac{1}{v^\gamma}\right)(1,t) = 0, \quad t>0 \\ v(x,0) = v_0(x), u(x,0) = u_0(x), 0 \le x \le 1. \end{cases}$$

where $\gamma > 1$ is a constant.

Prove that

(1)

$$\int_0^1 \left(\frac{1}{2}u^2 + \frac{1}{\gamma-1}v^{1-\gamma}\right)(x,t)dx \le \int_0^1 \left(\frac{1}{2}u_0^2 + \frac{1}{\gamma-1}v_0^{1-\gamma}\right)(x)dx, \quad \text{for} \quad t>0.$$

(2) there exist positive constants c_1, c_2, c_3, c_4 independent of x and t such that

$$c_1 v_0(x)\left(1 + c_2 v_0^{-\gamma}(x)t\right)^{\frac{1}{\gamma}} \le v(x,t) \le c_3 v_0(x)\left(1 + c_4 v_0^{-\gamma}(x)t\right)^{\frac{1}{\gamma}}$$

for $0 < x < 1$, $t > 0$.